COMBINATORICS TILING PROBLEM

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Abstract. Combinatorics is the field of mathematics studying the combination and permutation of sets of elements and the relationships that constitute their properties. A problem proposed in *The College Mathematics Journal* asks for a closed form expression for the number of ways to tile an n $\times$ n square with 1×1 squares and $(n-1) \times 1$ rectangles (each of which may be placed horizontally or vertically) for an integer $n \geq 3$. Using a 3 x 3 square as a starting point, we determined all of the possible cases by hand. Upon doing so, we were able to determine generalizable patterns for n cases and formulate combinations for each. Finally, using a series of identities we rewrote the formula into a compact form.

1. Introduction

Problem 1216 as proposed in *The College Mathematics Journal* [1]: For an integer $n \geq 3$, find a closed form expression for the number of ways to tile an $n \times n$ square with 1×1 squares and $(n-1) \times 1$ rectangles (each of which may be placed horizontally or vertically).

We begin by making two observations. First, the number of ways to tile an $n \times n$ square with k $(n-1) \times 1$ rectangles placed vertically and j $(n-1) \times 1$ rectangles placed horizontally is the same as the number of ways to tile an $n \times n$ square with k $(n-1) \times 1$ rectangles placed horizontally and j $(n-1) \times 1$ rectangles placed vertically assuming $k \neq j$. This can be seen by rotating each tiling by 90 degrees. The second observation is found in Lemma 1.

2. Solution

Lemma 1. *There does not exist an arrangement of k $(n-1) \times 1$ rectangles placed horizontally (vertically), $k \geq 3$, and j $(n-1) \times 1$ rectangles placed vertically (horizontally), $j \geq 2$, within an $n \times n$ square.*

Proof. Without loss of generality, place k $(n-1) \times 1$ rectangles horizontally and j $(n-1) \times 1$ rectangles vertically within an $n \times n$ square. Begin by placing the k horizontal tiles within the square. Note that each row of the square contains at most one such tile. When a row contains a horizontal tile, there is exactly one 1×1 square not covered by the tile.

At most one column will have $(n-1)$ or more 1×1 squares not covered by a horizontal tile. Suppose not. Then one such column will be created by lining up at least $k-1$ of the horizontal tiles to the left (or the right) side of the $n \times n$ square while the kth horizontal tile is either on the same side or at the top or the bottom of the square. Then any other column will have at most $(n-k+1)$ 1×1 squares not covered by the horizontal tiles. Since $k \geq 3$, $n-k+1 \leq n-2$ which means a second vertical tile cannot be placed within the $n \times n$ square. $\square$

Theorem 2. *We have determined that there are*

$$2(3^n) + 2^{n+3} + 2^{n+2} - 19$$

ways to tile an $n \times n$ square as described in Problem 1216.

Proof. We consider several cases based on the number of $(n-1)\times 1$ tiles placed within the $n \times n$ square.

Case 1. Suppose no $(n-1)\times 1$ tiles are used. Then there is one way to tile the $n \times n$ square with only 1×1 square tiles.

Case 2. Suppose k, $1 \le k \le n$, $(n-1)\times 1$ tiles are placed within the $n \times n$ square and all are placed horizontally or all are placed vertically. Then there are $\binom{n}{k}$ ways to choose the rows (columns) in which to place the horizontal (vertical) tiles. Each tile can be placed to the left or the right (top or bottom) of the $n \times n$ square. Therefore, there are $\sum_{k=1}^{n} 2^{k+1}\binom{n}{k}$ ways to tile the $n \times n$ square with all $(n-1)\times 1$ tiles being placed horizontally or all vertically.

Case 3. Suppose one $(n-1)\times 1$ tile is placed horizontally and one $(n-1)\times 1$ tile is placed vertically within the $n \times n$ square. If the horizontal tile is placed in the top row or the bottom row, there are $n+1$ options of where to place the vertical tile. Then there are $4(n+1)$ ways to place these two tiles in this case. If the horizontal tile is placed in neither the top nor the bottom row, then there are two options of where to place the vertical tile. This gives $4(n-2)$ ways to place these two tiles in this case. Therefore, there are $8n-4$ ways to tile the $n \times n$ square with one $(n-1)\times 1$ tile placed horizontally and one placed vertically.

Case 4. Suppose k, $2 \le k \le n$, $(n-1)\times 1$ tiles are placed horizontally (vertically) and one $(n-1)\times 1$ tile is placed vertically (horizontally). Without loss of generality, we assume the k tiles are placed horizontally and one tile is placed vertically.

Suppose all k horizontal tiles are placed along the same side of the $n \times n$ square. Then there are $2\binom{n}{k}$ ways to place the horizontal tiles and 2 ways to place the vertical tile. Thus, $4\binom{n}{k}$ ways to tile in this fashion.

Suppose a horizontal tile is placed in either the top row or the bottom row. If this is not the case, then all tiles are placed to the same side of the $n \times n$ square and has been counted previously. The remaining $k-1$ horizontal tiles are placed in the middle and on the opposite side of the $n \times n$ square. Then there is one option to place the vertical tile. Therefore, there are $4\binom{n-2}{k-1}$ ways to place the tiles in this fashion.

Suppose horizontal tiles are placed in both the top and bottom rows. They must be placed on opposite sides of the $n \times n$ square. The remaining $k-2$ horizontal tiles are placed in the middle rows and all are placed to the same side. There is then one option for placing the vertical tile. Therefore, there are $4\binom{n-2}{k-2}$ ways to place the tiles in this fashion.

We now add these totals and double the sum to find the total number of ways to tile the $n \times n$ square with k, $2 \le k \le n$, $(n-1)\times 1$ tiles in one direction and one $(n-1)\times 1$ tile in the other direction is $8\sum_{k=2}^{n}\left(\binom{n}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-2}\right)$ ways.

Case 5. Suppose two $(n-1)\times 1$ tiles are placed horizontally and 2 $(n-1)\times 1$ tiles are placed vertically. There are exactly two ways to placed the tiles in this fashion by placing them in the corners of the $n \times n$ square.

Therefore, the total is then

$$1 + \sum_{k=1}^{n} 2^{k+1}\binom{n}{k} + 8n - 4 + 8\sum_{k=2}^{n}\left(\binom{n}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-2}\right) + 2.$$

We now simplify this expression. Consider $\sum_{k=1}^{n} 2^{k+1}\binom{n}{k}$.
We can rewrite this as $2\sum_{k=1}^{n} 2^{k}\binom{n}{k} = 2(3^n - 1)$ using the well-known identity $\sum_{k=0}^{n} 2^{k}\binom{n}{k} = 3^n$, (see [2]).

We know $\sum_{k=2}^{n} \binom{n}{k} = 2^n - n - 1$ using the identity $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Using the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, (see [2]), we find that $\binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$. Then we see that $\sum_{k=2}^{n} \binom{n-1}{k-1} = 2^{n-1} - 1$ after shifting the index.

Now we have

$$
\begin{aligned}
&1 + \sum_{k=1}^{n} 2^{k+1}\binom{n}{k} + 8n - 4 + 8\sum_{k=2}^{n}\left(\binom{n}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-2}\right) + 2 \\
=&1 + 2(3^n) + 8n - 4 + 8(2^n - n - 1) + 8(2^{n-1} - 1) + 2 \\
=&2(3^n) + 2^{n+3} + 2^{n+2} - 19.
\end{aligned}
$$

□

3. Conclusion

Conclusively, by experimenting with a 3×3 square, we were able to come up with cases that could be generalized to an $n \times n$ square. Upon doing so, we wrote these results in terms of combinations which could then be simplified into a closed form expression using various identities. Upon completion of the problem we submitted our results to *The College Mathematics Journal* to be considered for publication.

4. Acknowledgement

The authors would like to thank the Swain Family School of Science and Mathematics and the Department of Mathematical Sciences at The Citadel for hosting the Math Research Experience Camp for Lowcountry High School Students in the Summer of 2022. The authors would also like to thank Ethan Curb and Cameron Moening for their contribution to the problem solution.

References

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[2] M. Spivey, *The Art of Proving Binomial Identities* (2019) Chapman & Hall.