PROOFS OF TWO FIBONACCI IDENTITIES

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Abstract. Fibonacci numbers are a recursive sequence that were originally used to model the breeding of rabbits under ideal circumstances. Fibonacci numbers are also connected to other topics of number theory like: binomial coefficients and Pascal's triangle.

We present solutions for two open problems from *The Fibonacci Quarterly*. Using known identities and knowledge of Fibonacci, we solved problems B-1301 and B-1305.

1. Introduction

The sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ is called the *Fibonacci sequence*. It was named in honor of Leonardo of Pisa —also known as Fibonacci— who gave the first applications of this sequence. In his book *Liber Abaci*, he introduced the Fibonacci sequence to the western world to model how fast rabbits could breed under ideal circumstances. The sequence was known to Indian authors as early as 450 BC, with Pingala's *Chandasastra* showing knowledge of the sequence.

The Fibonacci sequence has several applications in number theory, showing relations to Pascal's triangle and binomial coefficients. The sequence also has an intimate relationship with the golden ratio with the ratio of sequential Fibonacci numbers having a limit of the golden ratio.

The Fibonacci sequence, denoted by $\{F_n\}$, is a recursive sequence, meaning a value is determined by the previous two numbers of the sequence. It is defined by $F_n = F_{n-1} + F_{n-2}$, for all $n \geq 2$, with the initial conditions $F_0 = 0$ and $F_1 = 1$. Similarly, the Lucas sequence, denote by $\{L_n\}$, is recursively defined as $L_n = L_{n-1} + L_{n-2}$, for all $n \geq 2$, where $L_0 = 2$, and $L_1 = 1$. The first nine Lucas numbers are $2, 1, 3, 4, 7, 11, 18, 29$.

Fibonacci and Lucas numbers can be represented using a closed-form expression called Binet's formula:

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad L_n = \alpha^n + \beta^n, \quad \text{where} \quad \alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

In this paper, we solve two open problems proposed in the journal *The Fibonacci Quarterly*. Thus, we solved the problems B-1301 and B-1305 (see [3]).

2. Problem B-1301

In this section, we use $\lfloor x \rfloor$ to represent the floor of the real number x. This means the value of x rounded down to the nearest whole number. For example, $\lfloor 1.618 \rfloor = 1$.

2.1. **Problem** [3, B-1301]**.** Show that, for any integer $n \geq 0$,

$$\left\lfloor \sqrt{F_{2n+1}F_{2n+2}L_{2n+3}} \right\rfloor = F_{3n+3}. \tag{1}$$

2.2. **Solution.** We start this proof given an identity that we have proved using Binet's formula.

$$F_{2n+1}F_{2n+2}L_{2n+3} = F_{3n+3}^2 + 2F_nF_{n+1}. \tag{2}$$

After taking the floor of the square root of both sides of this previous equation, we have that the left-hand side is equal to that of (1). It can also be seen that the right-hand side of the equation (2) contains the term F_{3n+3}, which is also in the right side of the problem.

$$\left\lfloor \sqrt{F_{2n+1}F_{2n+2}L_{2n+3}} \right\rfloor = \left\lfloor \sqrt{F_{3n+3}^2 + 2F_nF_{n+1}} \right\rfloor.$$

Since $\sqrt{F_{3n+3}^2} = F_{3n+3}$, we must prove that $2F_nF_{n+1}$ is less than the distance to the next perfect square. Since it is known that the difference between any perfect square n^2 and the next perfect square is $2n+1$, we must prove that $2F_nF_{n+1} < 2F_{3n+3} + 1$.

Expanding F_{3n+3} and $2F_nF_{n+1}$, using Binet's formula, we can see that

$$F_{3n+3} = F_{n+1}\left(L_{2n+2} - (-1)^n\right) \text{ and } 2F_nF_{n+1} = \frac{2}{5}\left(L_{2n+1} - (-1)^n\right).$$

Accordingly, we can see that $2F_nF_{n+1} < 2F_{3n+3} - 1$. Because $2F_nF_{n+1}$ is less than the difference to the next perfect square, the decimal portion of $\sqrt{F_{3n+3}^2 + 2F_nF_{n+1}}$ is given by $2F_nF_{n+1}$. Since the decimal portion is removed by the floor, this means it is equal to F_{3n+3}, proving the original problem.

3. Problem B-1305

In this problem, the notation $\sum_{k=1}^{n} x$ is used to represent the sum of x from 1 to n. For example, $\sum_{k=1}^{3} k = 6$, because $1 + 2 + 3 = 6$.

Additionally, the terminology telescoping sum is used to refer to a sum where subsequent terms cancel out, leaving a closed form expression after simplification.

3.1. **Problem** [3, B-1305]**.** For any positive integer n, prove that

$$\sum_{k=1}^{n} F_{F_{3k}}F_{L_{3k}}L_{F_{3k}} = F_{F_{3n+1}}F_{F_{3n+2}} - 1. \tag{3}$$

3.2. **Solution.** We first look at an example for $n = 2$ to see that the left-hand side of (3) $F_{F_3}F_{L_3}L_{F_3} + F_{F_6}F_{L_6}L_{F_6} = F_2F_4L_2 + F_8F_{18}L_8 = 2550417$, and the right-hand side also equals $F_{F_7}F_{F_8} - 1 = F_{13}F_{21} - 1 = 2550417$.

In order to begin proving this, we will expand the sum on the left side to an arbitrary number of terms, in this case $n = 2$, to convert it to a telescoping sum, which can become a generalized closed-form expression.

The first identity we will use for the proof of the problem is $5F_nF_m = L_{n+m} - (-1)^m L_{n-m}$ (see for example, [2, 4]), on the two Fibonacci numbers in each term.

For this step, the identity can be written as $5F_nF_m = L_{n+m} - L_{n-m}$. Knowing that $m = L_{3n}$, it can be shown that this is always even because $L_{3n} = 2F_{3n-1} + F_{3n}$, due to the identity $L_n = 2F_{n-1} + F_n$, and it is already known that F_{3n} is always even. Thus, L_{3n} is the sum of two even numbers, so $(-1)^{L_{3n}}$ can be simplified to 1.

$$L_{F_3}(F_{F_3}F_{L_3}) + L_{F_6}(F_{F_3}F_{L_3}) = \frac{1}{5}L_{F_3}(L_{F_3+L_3} - L_{F_3-L_3}) + \frac{1}{5}L_{F_6}(L_{F_6+L_6} - L_{F_6-L_6}).$$

Before using our next identity, we distribute the Lucas number outside the parenthesis to the terms inside the parenthesis, making each term a product of Lucas numbers.

$$\frac{1}{5}(L_{F_3}L_{F_3+L_3} - L_{F_3}L_{F_3-L_3}) + \frac{1}{5}(L_{F_6}L_{F_6+L_6} - L_{F_6}L_{F_6-L_6}).$$

This grouping allows us to use the identity $L_mL_n = L_{n+m} + (-1)^m L_{n-m}$. Similarly, the expression $(-1)^m$ can be simplified to 1. m is always even because it is equal to $F_{3n} + L_{3n}$, making it the sum of two even numbers.

$$\frac{1}{5}((L_{2F_3+L_3} + L_{L_3}) - (L_{2F_3-L_3} + L_{-L_3})) + \frac{1}{5}((L_{2F_6+L_6} + L_{L_6}) - (L_{2F_6-L_6} - L_{-L_6})).$$

Next, the subscripts of the Lucas numbers with two terms can be simplified to one term using the identities $F_{3n+3} = 2F_{3n} + L_{3n}$ and $-F_{3n-3} = 2F_{3n} - L_{3n}$.

$$\frac{1}{5}((L_{F_6} + L_{L_3}) - (L_{-F_0} + L_{-L_3})) + \frac{1}{5}((L_{F_9} + L_{L_6}) - (L_{-F_3} + L_{-L_6})).$$

The numbers with negative subscripts can be changed to have positive subscripts using the identity $L_{-n} = (-1)^n L_n$ (see for example, [4]).

$$\frac{1}{5}((L_{F_6} + L_{L_3}) - (L_{F_0} + L_{L_3})) + \frac{1}{5}((L_{F_9} + L_{L_6}) - (L_{F_3} + L_{L_6})).$$

We can begin to combine like terms to reach the final closed form.

$$\frac{1}{5}(L_{F_6} + L_{L_3} - L_{F_0} - L_{L_3} + L_{F_9} + L_{L_6} - L_{F_3} - L_{L_6}) = \frac{1}{5}(L_{F_9} + L_{F_6} - L_{F_0}).$$

To show that this is equal to the right-hand side of the equation, we can use the first identity, $5F_nF_m = L_{n+m} - (-1)^m L_{n-m}$, on the expression $F_{F_{3n+1}}F_{F_{3n+2}} - 1$. Since m is of the form F_{3n+1}, and F_{3n} is always an even number, it is known that m will always be odd.

$$\frac{1}{5}(L_{F_{3n+2}+F_{3n+1}} + L_{F_{3n+2}-F_{3n+1}}) - 1.$$

Using the recursive sequence that defines Fibonacci numbers, $F_{n+2} = F_{n+1} + F_n$, the subscripts can be simplified, to obtain

$$\frac{1}{5}(L_{F_{3n+3}} + L_{F_{3n}}) - 1.$$

4. Conclusion

Problem B-1301 and Problem B-1305 were solved during June 2022 and submitted to *Fibonacci Quarterly* during the Math Research Experience Camp for Lowcountry High School students hosted by The Citadel.

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