Problem 1216 Proof

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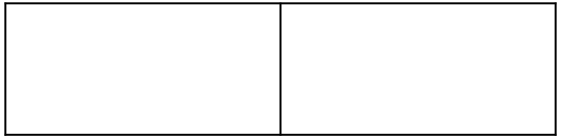
Problem Statement

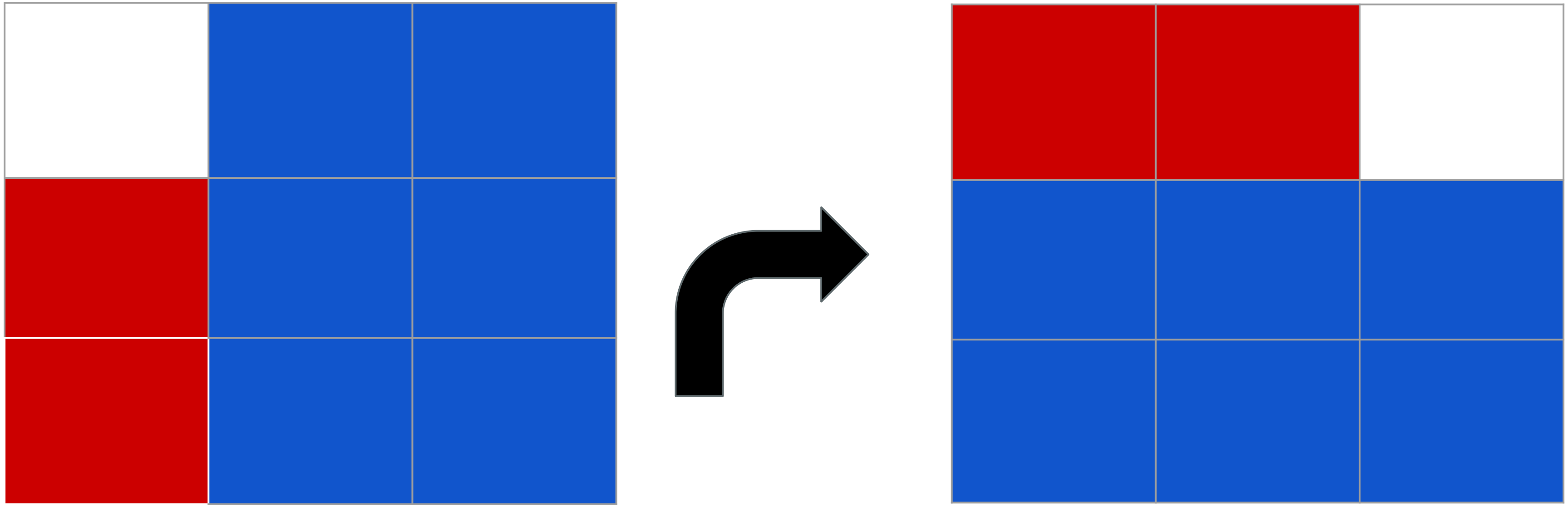
PROBLEM 1216

For an integer $n \geq 3$, find a closed form expression for the number of ways to tile an $n \times n$ square with 1×1 squares and $(n-1) \times 1$ rectangles (each of which may be placed horizontally or vertically).



Proposed by Oluwatobi Alabi, Government Science Secondary School Pyakasa Abuja, Abuja Nigeria.

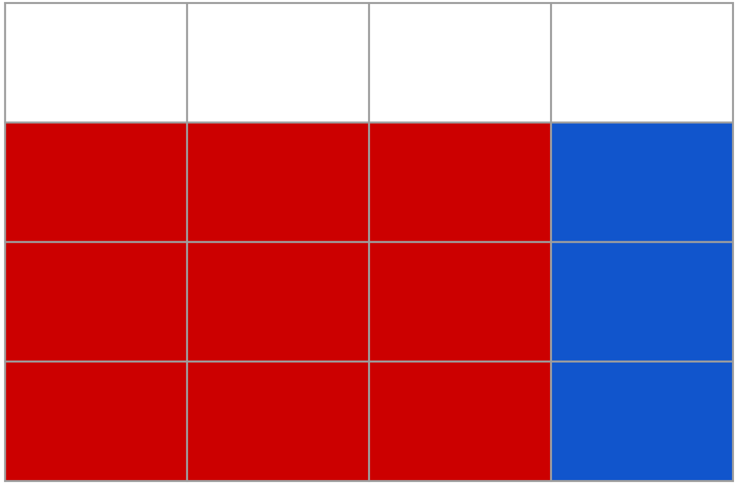
Initial Observations



The number of ways to tile an $n \times n$ square with k $(n - 1) \times 1$ rectangles placed vertically and j $(n - 1) \times 1$ rectangles placed horizontally is the same as the number of ways to tile an $n \times n$ square with k $(n - 1) \times 1$ rectangles placed horizontally and j $(n - 1) \times 1$ rectangles placed vertically assuming $k \neq j$.

Lemma

Lemma 1. *There does not exist an arrangement of k $(n-1)\times 1$ rectangles placed horizontally (vertically), $k \geq 3$, and j $(n-1)\times 1$ rectangles placed vertically (horizontally), $j \geq 2$ within an $n \times n$ square.*



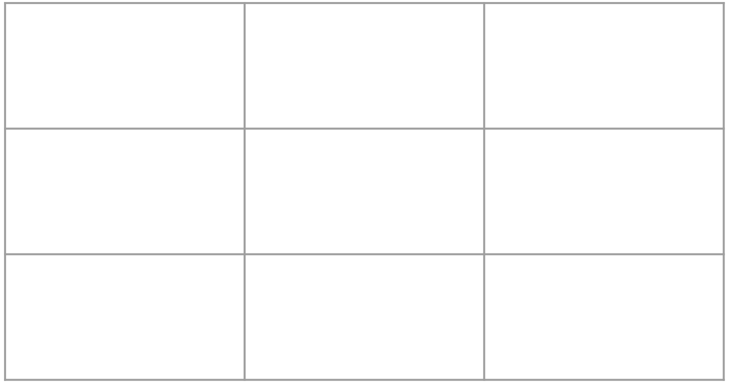
Combinations

Formula

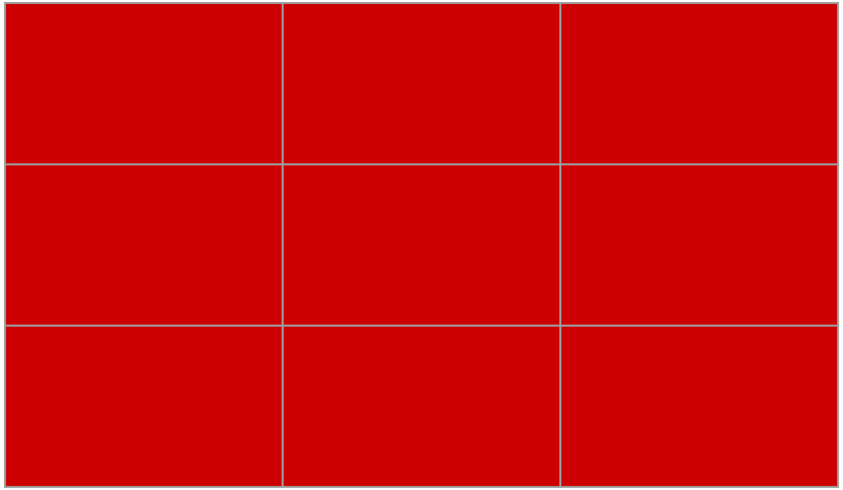
$$_nC_r = \frac{n!}{r!(n-r)!} = \binom{n}{k}$$

Problem Solution Approach - Cases

In solving this problem, it was crucial to consider all the possible cases in order to reach a generalization for an *n* x *n* squares. We began by taking a 3 x 3 square, as this is the smallest case in which there is a significant number of possible cases. Along with the lemma and our initial observation, we were able to eliminate a number of possible combinations and determine all of the cases for an *n* x *n* square.

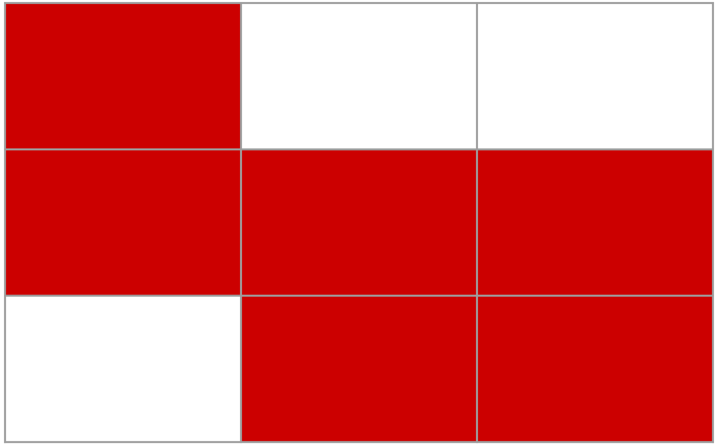


Case #1



If no (n - 1) x 1 tiles are used, there is only way to tile the n x n square with only 1 x 1 square tiles.

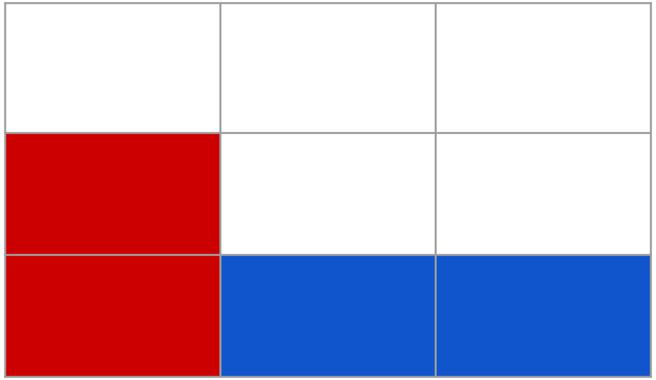
Case #2



Suppose we only have vertical/horizontal tiles. This leaves us *n* ways to choose the rows or columns for *k* tiles. The tiles can also be shifted up and down or left and right. The generalization of this case results in the following number of possibilities to tile the square with all (n -1) x 1 tiles being placed vertically or horizontally:

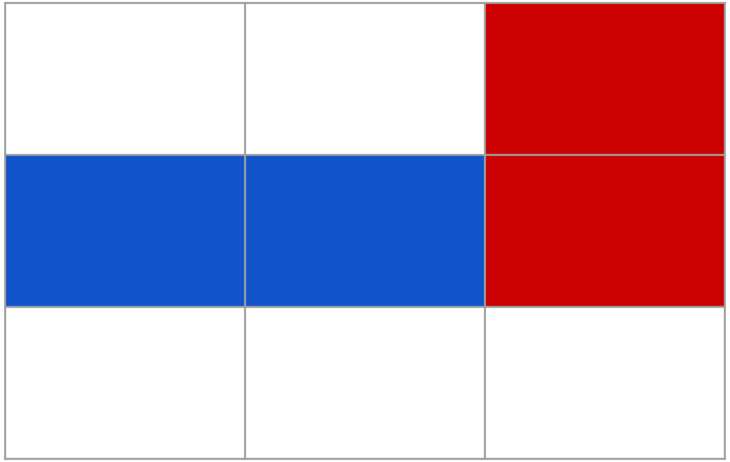
$$\sum_{k=1}^{n} 2^{k+1} \binom{n}{k}$$

Case #3 (Part 1)



Suppose we have one vertical and one horizontal tile. If the horizontal tile is placed in the top or bottom row, than there are $n + 1$ ways to place the vertical tile. As such, there are $4(n + 1)$ ways to place the two tiles in this case.

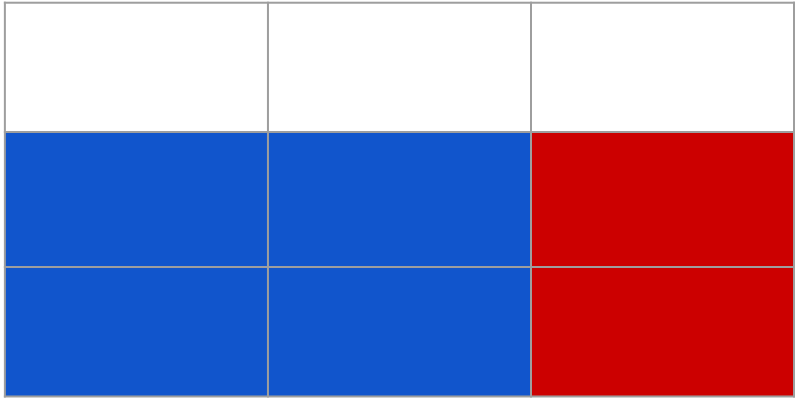
Case #3 (Part 2)



Suppose that the horizontal tile is placed in neither the top nor the bottom row. This leaves 2 options to place the vertical tile. As such, there are 4(*n*-2) ways to place the two tiles in this case. Combining this with the result from Part 1, we get the following number of ways to tile the square with 1 (*n*-1) x 1 tile placed vertically and 1 (*n*-1)x1 tile placed horizontally:

$$8n - 4$$

Case #4 (Part 1)



Suppose $2 \leq k \leq n$ tiles are placed horizontally and one tile is placed vertically. If all horizontal tiles are placed on the same side of the square, this leaves 2 times n choose k ways to place the k horizontal tiles and 2 ways to place the vertical tiles. As such, there are the following number of ways to place the tiles in this case:

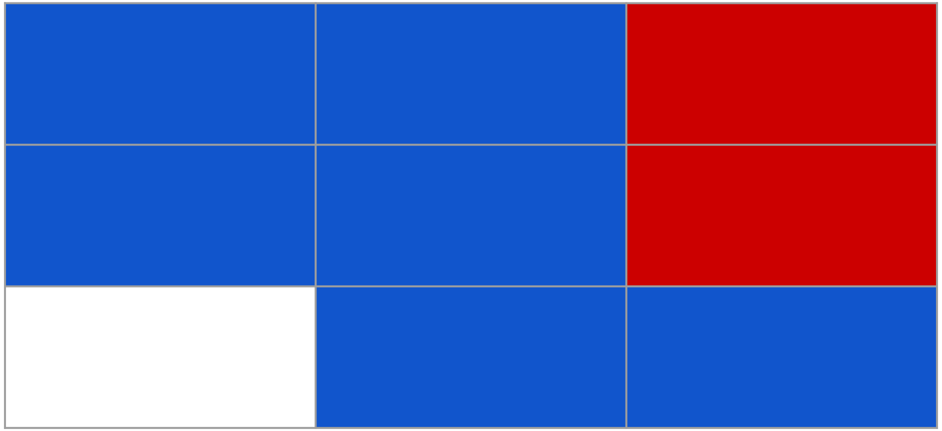
$$4\binom{n}{k}$$

Case #4 (Part 2)

Suppose a horizontal tile is placed in the top or bottom row and the remaining *k-1* tiles are placed in the middle on the opposite side. Then there is only one option to place the vertical tile. As such, there are the following number of ways to place the tiles in this case:

$$4\binom{n-2}{k-1}$$

Case #4 (Part 3)



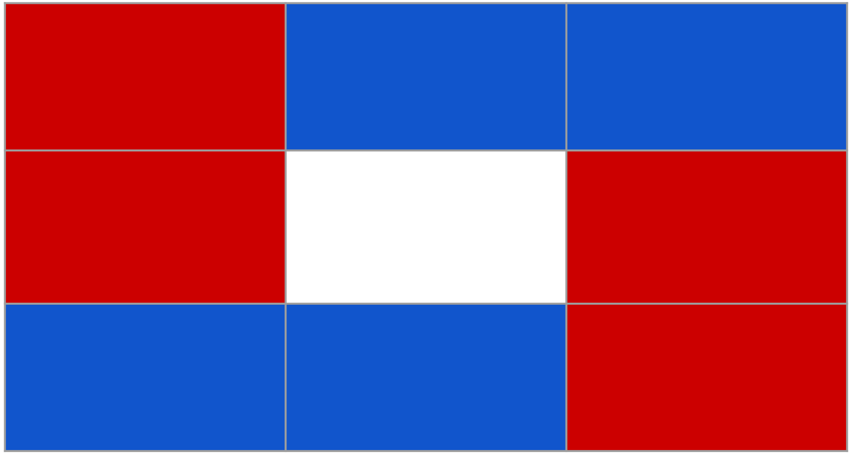
Suppose horizontal tiles are placed in both the top and bottom rows on opposite sides. The remaining *k* - 2 tiles are placed in the middle row(s) on the same side. This leaves only one option to place a vertical tile. As such, there are the following number of options to place the tiles in this case:

$$4\binom{n-2}{k-2}$$

Case #4 (Final)

We now add these totals and double the sum to find the total number of ways to tile the $n \times n$ square with k, $2 \leq k \leq n$, $(n-1) \times 1$ tiles in one direction and one $(n-1) \times 1$ tile in the other direction is $8 \sum_{k=2}^{n} \left(\binom{n}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-2} \right)$ ways.

Case #5



Suppose 2 (n - 1) x 1 tiles are placed vertically and 2 (n - 1) x 1 are placed horizontally. Then, there are exactly 2 ways to placed the tiles in this fashion by placing them in the corners of the square.

Combing the Cases

Summing all of the above cases gives the following result:

$$1 + \sum_{k=1}^{n} 2^{k+1}\binom{n}{k} + 8n - 4 + 8\sum_{k=2}^{n}\left(\binom{n}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-2}\right) + 2.$$

Simplifying this expression results in:

$$1 + 2(3^n) + 8n - 4 + 8(2^n - n - 1) + 8(2^{n-1} - 1) + 2$$

Further algebraic simplification gives:

$$2^{n+2} + 2^{n+3} + 2(3^n) - 19$$

Conclusion & Theorem

Theorem 2. *We have determined that there are*

$$2(3^n) + 2^{n+3} + 2^{n+2} - 19$$

ways to tile an $n \times n$ square as described in the problem.

Dimension (*n* x *n*)	# of way to tile
n = 3	131
n = 4	335
n = 5	851
n = 6	2207

Thank you!

Simplification - Proof

Known results:

1. $(x+y)^n = \sum_{k+1}^{n} \binom{n}{k} x^k y^{n-k}$
2. $3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$
3. $2^n = \sum_{k=0}^{n} \binom{n}{k}$
4. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Simplification - Proof

Using the result from 2, we can show $\sum_{k=1}^{n} 2^{k+1}\binom{n}{k} = \sum_{k=0}^{n} 2^{k+1}\binom{n}{k} - 2\binom{n}{0} = 2(3^n - 1)$

Using the result from 3, we can show $\sum_{k=2}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} - \binom{n}{0} - \binom{n}{1} = 2^n - 1 - n$

Using the result from 4, we can show $\sum_{k=2}^{n} \left(\binom{n-2}{k-1} + \binom{n-2}{k-2}\right) = \sum_{k=2}^{n} \binom{n-1}{k-1}$

Index shifting gives $\sum_{k=1}^{n-1} \binom{n-1}{k-1} - \binom{n-1}{0} + \binom{n-1}{n-1} = \sum_{k=1}^{n-1} \binom{n-1}{k-1}$

Letting k= $m+1$ gives $\sum_{m+1=1}^{m+1=n-1} \binom{n-1}{m} = \sum_{m=0}^{n-2} \binom{n-1}{m} = \sum_{m=0}^{n-1} \binom{n-1}{m} - \binom{n-1}{n-1}$

Using the result from 2, we can show $\sum_{m=0}^{n-1} \binom{n-1}{m} = 2^{n-1} - 1$

Simplification - Proof

Combining the above results gives:

$$1+\sum_{k=1}^{n} 2^{k+1}\binom{n}{k}+8n-4+8\sum_{k=2}^{n}\left(\binom{n}{k}+\binom{n-2}{k-1}+\binom{n-2}{k-2}\right)+2=2(3^n-1)+8(2^n+2^{n-1}-n)+8n-1$$

Further algebraic simplification gives:

$$2^{n+2}+2^{n+3}+2(3^n)-19$$