PROBLEM B-1305

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PROBLEM

- For any positive integer n, prove that

$$\sum_{k=1}^{n} F_{F_{3k}} F_{L_{3k}} L_{F_{3k}} = F_{F_{3n+1}} F_{F_{3n+2}} - 1$$

IDENTITIES

$$\begin{aligned} 5F_nF_m &= L_{n+m} - (-1)^m L_{n-m} \\ L_mL_n &= L_{n+m} + (-1)^m L_{n-m} \\ F_{3n+3} &= 2F_{3n} + L_{3n} \\ -F_{3n-3} &= 2F_{3n} - L_{3n} \end{aligned}$$

PROOF

- We will be expanding the summation to reach a generalizable closed-form expression.

- Since the summation can be expanded for any value n, we will be using $n = 1$ for this presentation.

STEP 1

$$\sum_{k=1}^{n} F_{F_{3k}} F_{L_{3k}} L_{F_{3k}}$$

- Since $n = 1$, this becomes

$$F_{F_3} F_{L_3} L_{F_3}$$

STEP 2

$$L_{F_3}(F_{F_3}F_{L_3})$$

- Identity 1 is

$$5F_nF_m = L_{n+m} - (-1)^m L_{n-m}$$

- After using identity 1, this becomes

$$\frac{1}{5}L_{F_3}(L_{F_3+L_3} - L_{F_3-L_3})$$

STEP 3

$$\frac{1}{5}L_{F_3}(L_{F_3+L_3} - L_{F_3-L_3})$$

- Distribute

$$\frac{1}{5}(L_{F_3}L_{F_3+L_3} - L_{F_3}L_{F_3-L_3})$$

STEP 4

$$\frac{1}{5}(L_{F_3}L_{F_3+L_3} - L_{F_3}L_{F_3-L_3})$$

- Identity 2 is

$$L_mL_n = L_{n+m} + (-1)^m L_{n-m}$$

- After using identity 2, this becomes

$$\frac{1}{5}((L_{2F_3+L_3} + L_{L_3}) - (L_{2F_3-L_3} + L_{-L_3}))$$

STEP 5

$$\frac{1}{5}((L_{2F_3+L_3} + L_{L_3}) - (L_{2F_3-L_3} + L_{-L_3}))$$

- Identities 3 and 4 are

$$\begin{aligned} F_{3n+3} &= 2F_{3n} + L_{3n} \\ -F_{3n-3} &= 2F_{3n} - L_{3n} \end{aligned}$$

- After using these identities, this becomes

$$\frac{1}{5}((L_{F_6} + L_{L_3}) - (L_{F_0} + L_{-L_3}))$$

STEP 6

$$\frac{1}{5}(L_{F_6} + L_{L_3} - L_{F_0} - L_{-L_3})$$

- The identity defining Lucas numbers with negative subscripts is

$$L_{-n} = (-1)^n L_n$$

- After using this identity, this becomes

$$\frac{1}{5}(L_{F_6} + L_{L_3} - L_{F_0} - L_{L_3})$$

STEP 7

$$\frac{1}{5}(L_{F_6} - L_{F_0} + L_{L_3} - L_{L_3})$$

- The terms L_{L_3} and $-L_{L_3}$ cancel out

$$\frac{1}{5}(L_{F_6} - L_{F_0})$$

STEP 8

- For any value of n, this can be generalized to a closed form expression

$$\frac{1}{5}(L_{F_{3n+3}} + L_{F_{3n}}) - 1$$

- Using identity 1, the other side of the equation can be transformed to

$$F_{F_{3n+1}} F_{F_{3n+2}} = \frac{1}{5}(L_{F_{3n+3}} + L_{F_{3n}}) - 1$$

GENERAL EXPRESSION

- The same cancellation applies for any value of n to reach the same expression

$$(F_{F_3}F_{L_3})L_{F_3} + (F_{F_6}F_{L_6})L_{F_6} + (F_{F_9}F_{L_9})L_{F_9}$$

$$\frac{1}{5}((L_{F_3+L_3} - L_{F_3-L_3})L_{F_3} + (L_{F_6+L_6} - L_{F_6-L_6})L_{F_6} + (L_{F_9+L_9} - L_{F_9-L_9})L_{F_9})$$

$$\frac{1}{5}((L_{F_3}L_{F_3+L_3} - L_{F_3}L_{F_3-L_3}) + (L_{F_6}L_{F_6+L_6} - L_{F_6}L_{F_6-L_6}) + (L_{F_9}L_{F_9+L_9} - L_{F_9}L_{F_9-L_9}))$$

GENERAL EXPRESSION (CONT)

$$\frac{1}{5}((L_{2F_3+L_3} + L_{L_3}) - (L_{2F_3-L_3} + L_{-L_3}) + (L_{2F_6+L_6} + L_{L_6}) - (L_{2F_6-L_6} + L_{-L_6}) + (L_{2F_6+L_6} + L_{L_6}) - (L_{2F_6-L_6} + L_{-L_6}))$$

$$\frac{1}{5}(L_{F_6} + L_{L_3} - L_{F_0} - L_{L_3} + L_{F_9} + L_{F_6} - L_{F_3} - L_{L_6} + L_{F_12} + L_{L_9} - L_{F_6} - L_{L_9})$$

$$\frac{1}{5}(L_{F_12} + L_{F_9} - L_{F_3} - L_{F_0})$$

THANK YOU