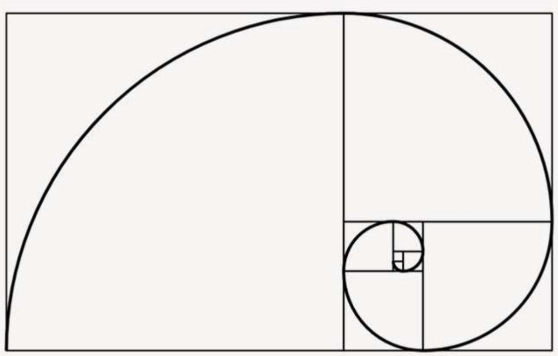
Problem B-1301: A Proof

Ethan Curb and Lavender Milligan



Intro to Fibonacci and Lucas Numbers

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

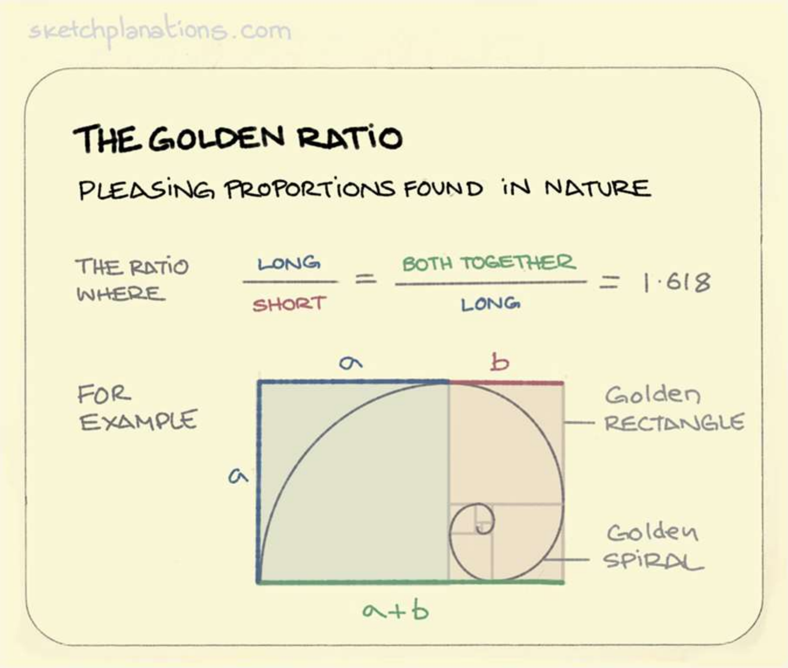
$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

The above definitions for α and β are vital to understanding Binet's Formula.

Fibonacci Numbers in Nature and the Golden Ratio



The Problem: B-1301 (Fibonacci Quarterly)

Show that, for any integer $n \geq 0$,

$$\lfloor \sqrt{F_{2n+1}F_{2n+2}L_{2n+3}} \rfloor = F_{3n+3}.$$

Binet's Formula

Binet's Formula plays an important role in solving this problem.

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$$

The Proof:

Proved using Binet's Formula.

Using Binet's Formula, we have determined that

$$F_{2n+1}F_{2n+2}L_{2n+3} = F_{3n+3}^2 + 2F_nF_{n+1}.$$

And so we can see that

$$\left\lfloor \sqrt{F_{2n+1}F_{2n+2}L_{2n+3}} \right\rfloor = \left\lfloor \sqrt{F_{3n+3}^2 + 2F_nF_{n+1}} \right\rfloor.$$

Explanation to follow.

$$(\frac{\alpha^{2n+1}-\beta^{2n+1}}{\sqrt{5}})(\frac{\alpha^{2n+2}-\beta^{2n+2}}{\sqrt{5}})(\frac{\alpha^{2n+3}+\beta^{2n+3}}{1}) = F_{2n+1}F_{2n+2}L_{2n+3}$$

$$(\frac{\alpha^{3n+3}-\beta^{3n+3}}{\sqrt{5}})^2 + 2(\frac{\alpha^{n}-\beta^{n}}{\sqrt{5}})(\frac{\alpha^{n+1}-\beta^{n+1}}{\sqrt{5}}) = F_{3n+3}^2 + 2F_nF_{n+1}$$

Through expansion, we can see that

$$F_{2n+1}F_{2n+2}L_{2n+3} = F_{3n+3}^2 + 2F_nF_{n+1}.$$

Proving the Floor

Proved using Binet's Formula and the relationship between Fibonacci and Lucas numbers.

Using Binet's Formula again, we find that

$$F_{3n+3} = F_{n+1}(L_{2n+2} - (-1)^n) > 2(L_{2n+1} - (-1)^n)/5 = 2F_nF_{n+1}.$$

Therefore,

$$F_{3n+3} > 2F_nF_{n+1}.$$

Proving the Floor (Cont.)

We have now determined that $F_{3n+3} > 2F_nF_{n+1}$.

This implies that the decimal part of $\sqrt{F_{3n+3}^2 + 2F_nF_{n+1}}$ is given by $2F_nF_{n+1}$.

Ergo, $\lfloor\sqrt{F_{2n+1}F_{2n+2}L_{2n+3}}\rfloor = F_{3n+3}$.

QED.

Thanks!